7.1 Integration by Parts

The product rule states that if $\boldsymbol{f}$ and $\boldsymbol{g}$ are differentiable functions, then

$$\frac{d}{dx}[f(x)\cdot g(x)] = f(x)g'(x) + f'(x)g(x)$$

Taking the indefinite integral gives:

$$f(x)g(x) = \int [f(x)g'(x) + f'(x)g(x)]dx$$

$$f(x)g(x) = \int (f(x)g'(x))dx + \int (f'(x)g(x))dx$$

We can rearrange this equation to be:

$$\int (\boldsymbol{f}(\boldsymbol{x})\boldsymbol{g}'(\boldsymbol{x}))\boldsymbol{dx} = \boldsymbol{f}(\boldsymbol{x})\boldsymbol{g}(\boldsymbol{x}) - \int (\boldsymbol{f}'(\boldsymbol{x})\boldsymbol{g}(\boldsymbol{x}))\boldsymbol{dx}$$

This new formula is called Integration by Parts. Also if we let **u = f(x)** and **v = g(x)** and differentiate we get: **du = f '(x)dx** and **dv = g '(x)dx**

The formula for **Integration by Parts** becomes:

$$\int \boldsymbol{u}\cdot\boldsymbol{dv} = \boldsymbol{u}\cdot\boldsymbol{v} - \int \boldsymbol{v}\cdot\boldsymbol{du}$$

Example:

$$\int xe^x dx$$

$$Let\ \boldsymbol{u} = \boldsymbol{x}\ and\ \boldsymbol{v} = \ \boldsymbol{e^x}$$

$$then\ \boldsymbol{du} = \boldsymbol{dx}\ \ and\ \ \boldsymbol{dv} = \ \boldsymbol{e^x dx}$$

Substitute this into the Integration by Parts formula:

$$\int u\cdot dv = u\cdot v - \int v\cdot du$$

$$\int xe^x dx = xe^x - \int e^x dx$$

$$\int \overset{u}{x}\,\overset{dv}{e^x dx} = \overset{u\cdot v}{\boldsymbol{xe^x}} - \overset{v\ \ du}{\boldsymbol{e^x}} + \boldsymbol{C}$$

Example: (Sometimes we have to use integration by parts more than once.)

$$\int (x^2+2x)\,(\cos(x))dx$$

$$Let\ \boldsymbol{u} = \ x^2 + 2x\quad and\quad \boldsymbol{dv} = \cos(x)dx$$

$$then\ \boldsymbol{du} = (2x+2)dx\ \ and\ \ \boldsymbol{v} = \sin(x)$$

(We have to choose one of the factors to be **u** and let the other factor be **dv**.)

$$\int (x^2+2x)\,(\cos(x))dx = (x^2+2x)\cdot\sin(x) - \int \sin(x)(2x+2)dx$$

(Solve by this by Integration of Parts also)

u = 2x + 2 dv = sin(x)

du = 2dx v = −cos(x)

$$\int (x^2+2x)(\cos(x))dx = (x^2+2x)\sin(x) - \left[(2x+2)\cdot -\cos(x) - \int 2(-\cos(x))dx\right]$$
$$= \boldsymbol{(x^2+2x)\sin(x) + (2x+2)\cos(x) - 2\sin(x) + C}$$

Now, if we combine the formula for integration by parts with Part 2 of the Fundamental Theorem of Calculus, we can evaluate definite integrals by parts.

$$\boldsymbol{\int_a^b f(x)g'(x)dx = f(x)g(x)]_a^b - \int_a^b g(x)f'(x)dx}$$

Example: Evaluate:

$$\int_1^2 \ln(x)dx$$

Let $\mathbf{u} = \ln(x)$ and $\boldsymbol{dv = dx}$

then $\boldsymbol{du = \frac{1}{x}dx}$ and $\boldsymbol{v = x}$

$$\int_1^2 \ln(x)dx = \ln(x)\cdot x]_1^2 - \int_1^2 x\cdot\frac{1}{x}dx$$
$$= \ln(x)\cdot x - x]_1^2$$
$$= (\ln(2)\cdot 2 - 2) - (\ln(1)\cdot 1 - 1)$$
$$= 2\ln(2) - 2 + 1$$
$$= \mathbf{2\ln(2) - 1}$$

Note: Sometimes it can be tricky in deciding which factor to be **u** and **dv.** If you try one way and it doesn't work, the switch the **u** and **dv** and try again.